PADERBORN MINICOURSE

1. Lecture 1: Lie Groups, Lie Algebras, and Homogeneous Spaces

1.1. **Subsets of $\mathbb{R}^n$ with symmetry.** In mathematics, we often study subsets of $X \subset \mathbb{R}^n$ or subsets $Y \subset \mathbb{C}^n$.

Example 1.1. The n-sphere $\mathbb{S}^n \subset \mathbb{R}^n$ is defined to be the set of real solutions to the single polynomial equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

We may also view $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ as the solution to the single polynomial equation

$$z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = 1.$$

Example 1.2. The real (p,q)-hyperboloid $\mathcal{H}_{p,q} \subset \mathbb{R}^n$ $(n = p+q)$ is defined to be the set of real solutions to the single polynomial equation

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2 = 1.$$

The complex (p,q)-hyperboloid $\mathcal{H}_{p,q}^{\mathbb{C}} \subset \mathbb{C}^n$ $(n = p+q)$ is defined to be the set of complex solutions to the single polynomial equation

$$z_1\overline{z}_1 + \cdots + z_p\overline{z}_p - z_{p+1}\overline{z}_{p+1} - \cdots - z_n\overline{z}_n = 1.$$

Next, if $X \subset \mathbb{R}^n$ or $Y \subset \mathbb{C}^n$ is a subset, then we may consider the group G_X of real linear symmetries of X and the group G_Y of complex linear symmetries of Y. More precisely, define

$$G_X := \{g \in \mathrm{GL}(n,\mathbb{R}) \mid g \cdot v \in X \text{ whenever } v \in X\}$$

if $X \subset \mathbb{R}^n$ and

$$G_Y := \{g \in \mathrm{GL}(n,\mathbb{C}) \mid g \cdot v \in Y \text{ whenever } v \in Y\}$$

if $Y \subset \mathbb{C}^n$.

Exercise 1.3. (a) Define $\mathrm{O}(n) \subset \mathrm{GL}(n,\mathbb{R})$ to be the collection of matrices A such that $A^TA = I$. We call $\mathrm{O}(n)$ the *orthogonal group*. Show that $\mathrm{O}(n)$ is the group of linear symmetries of the n-sphere $\mathbb{S}^n \subset \mathbb{R}^n$.

(b) Define $\mathrm{U}(n) \subset \mathrm{GL}(n,\mathbb{C})$ to be the collection of matrices A such that $A^*A = I$. We call $\mathrm{U}(n)$ the *unitary group*. Show that $\mathrm{U}(n)$ is the group of (complex) linear symmetries of the $2n-1$ sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$.

(c) Define

$$I_{p,q} := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

where the n by n matrix $I_{p,q}$ has p ones along the diagonal, q minus ones along the diagonal, zeroes in all non-diagonal entries, and $n = p+q$. Then define $\mathrm{O}(p,q) \subset \mathrm{GL}(n,\mathbb{R})$ $(n = p+q)$ to be the collection of matrices A such that $A^T I_{p,q} A = I_{p,q}$. We call $\mathrm{O}(p,q)$ the *indefinite orthogonal group* with signature (p,q). Show that $\mathrm{O}(p,q)$ is the group of linear symmetries of the (real) hyperboloid $\mathcal{H}_{p,q} \subset \mathbb{R}^n$.

(d) Define $\mathrm{U}(p,q) \subset \mathrm{GL}(n,\mathbb{C})$ $(n = p+q)$ to be the collection of matrices A such that $A^* I_{p,q} A = I_{p,q}$. We call $\mathrm{U}(p,q)$ the *indefinite unitary group* with signature (p,q). Show that $\mathrm{U}(p,q)$ is the group of (complex) linear symmetries of the (complex) hyperboloid $\mathcal{H}^{\mathbb{C}}_{p,q} \subset \mathbb{C}^n$.

If one takes a generic subset $X \subset \mathbb{R}^n$ or $Y \subset \mathbb{C}^n$, then there may be no nontrivial linear symmetries of X or Y. However, some of the most interesting subsets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{C}^n$ have lots of linear symmetries. In these notes, we will assume that X (respectively Y) is a *homogeneous space* for G_X (resp. G_Y). This means that for every $x, y \in X$, there exists $g \in G_X$ such that $g \cdot x = y$.

Exercise 1.4. (a) Show that $\mathbb{S}^n \subset \mathbb{R}^n$ is a homogeneous space for the group $\mathrm{O}(n)$ and $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ is a homogeneous space for the group $\mathrm{U}(n)$.

(b) (Harder) Show that $\mathcal{H}_{p,q} \subset \mathbb{R}^n$ $(n = p+q)$ is a homogeneous space for the group $\mathrm{O}(p,q)$ and $\mathcal{H}^{\mathbb{C}}_{p,q} \subset \mathbb{C}^n$ $(n = p+q)$ is a homogeneous space for the group $\mathrm{U}(p,q)$.

In this series of lectures, we are interested in studying analysis on spaces $X \subset \mathbb{R}^n$ (or $Y \subset \mathbb{C}^n$) that are homogeneous spaces for the corresponding linear group of symmetries G_X (or G_Y). Our first step is to better understand linear subgroups $G \subset \mathrm{GL}(n,\mathbb{R})$ (and linear subgroups $G \subset \mathrm{GL}(n,\mathbb{C})$).

1.2. **Linear Groups, Lie Algebras, and the Exponential Map.**

Let $\mathfrak{gl}(n,\mathbb{R})$ (resp. $\mathfrak{gl}(n,\mathbb{C})$) denote the collection of all n by n matrices with real (resp. complex) entries. The *exponential map*

$$\exp\colon \mathfrak{gl}(n,\mathbb{R}) \longrightarrow \mathrm{GL}(n,\mathbb{R})$$

is defined to be

$$\exp X := \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

The same formula defines the exponential map

$$\exp\colon \mathfrak{gl}(n,\mathbb{C}) \longrightarrow \mathrm{GL}(n,\mathbb{C}).$$

Exercise 1.5. (a) Show $\exp X$ converges for all $X \in \mathfrak{gl}(n,\mathbb{R})$ and $X \in \mathfrak{gl}(n,\mathbb{C})$.
(b) If $S \in \mathrm{GL}(n,\mathbb{C})$, show $\exp(SXS^{-1}) = S\exp(X)S^{-1}$.
(c) If $\lambda_1,\dots,\lambda_n$ are the (generalized) eigenvalues of $X \in \mathfrak{gl}(n,\mathbb{C})$, show that $e^{\lambda_1},\dots,e^{\lambda_n}$ are the (generalized) eigenvalues of $\exp X \in \mathrm{GL}(n,\mathbb{C})$. (Hint: Utilize Jordan normal form and part (b)).
(d) Conclude $e^{\mathrm{tr}(X)} = \det(\exp(X))$, and conclude $\exp X \in \mathrm{GL}(n,\mathbb{R})$ (resp. $\exp X \in \mathrm{GL}(n,\mathbb{C})$) for all matrices $X \in \mathfrak{gl}(n,\mathbb{C})$ (resp. $X \in \mathfrak{gl}(n,\mathbb{C})$).

Definition 1.6. A *real linear group* is a closed subgroup $G \subset \mathrm{GL}(n,\mathbb{R})$. A *complex linear group* is a closed subgroup $G \subset \mathrm{GL}(n,\mathbb{C})$.

Definition 1.7. The *Lie algebra* of a real linear group $G \subset \mathrm{GL}(n,\mathbb{R})$ (resp. complex linear group $G \subset \mathrm{GL}(n,\mathbb{C})$, denoted $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ (resp. $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$) is the collection of $X \in \mathfrak{gl}(n,\mathbb{R})$ (resp. $X \in \mathfrak{gl}(n,\mathbb{C})$) such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$. If $X \in \mathfrak{g}$, we call the collection of $\exp(tX)$ with $t \in \mathbb{R}$ the *one-parameter subgroup* of G corresponding to X.

Exercise 1.8. (a) Show that the Lie algebra of $\mathrm{O}(n)$ is $\mathfrak{o}(n)$, the collection of $X \in \mathfrak{gl}(n,\mathbb{R})$ satisfying $X^T + X = 0$. This is the collection of skew symmetric matrices.
(Hint: Differentiate the condition $\exp(tX)^T\exp(tX) = I$ and evaluate at $t = 0$).
(b) Show that the Lie algebra of $\mathrm{U}(n)$ is $\mathfrak{u}(n)$, the collection of $X \in \mathfrak{gl}(n,\mathbb{C})$ satisfying $X^* = -X$. This is the collection of skew Hermitian matrices.

In order to study a linear group G, it is very useful to study its Lie algebra $\mathfrak{g}$. There are many examples of this phenomenon; let us first mention a fundamental one.

Proposition 1.9. *(1) The Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ (resp. $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$) is a vector subspace.*
(2) Every closed linear group $G \subset \mathrm{GL}(n,\mathbb{R})$ or $G \subset \mathrm{GL}(n,\mathbb{C})$ is a smooth manifold.

(3) *Let* $e \in G$ *denote the identity element. Then there exist an open subset* $0 \in \Omega \subset \mathfrak{g}$ *and an open subset* $e \in \Xi \subset G$ *such that* $\exp \colon \Omega \to \Xi$ *is a diffeomorphism.*

We omit the proofs. See for instance the introduction of [Kna05]. Part (c) of this proposition tells us that the Lie algebra together with the exponential map give a coordinate chart for G near the identity element $e \in G$. In particular, we may identify $\mathfrak{g} \simeq T_eG$. We remark that this is actually how you prove part (b), that G is a smooth manifold.

Exercise 1.10. (a) Show that $\iota_{\mathbb{C}} \colon \mathrm{GL}(m, \mathbb{C}) \hookrightarrow \mathrm{GL}(2m, \mathbb{R})$ is a linear group. Moreover, show that it is the subgroup of matrices that commute with m by m block matrix

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

(b) Show that the Lie algebra of $\mathrm{GL}(m, \mathbb{C})$ is the image of $\iota_{\mathbb{C}} \colon \mathfrak{gl}(m, \mathbb{C}) \subset \mathfrak{gl}(2m, \mathbb{R})$, the collection of all $2m$ by $2m$ real matrices commuting with J.

(c) Check that $\exp(\iota_{\mathbb{C}}(X)) = \iota_{\mathbb{C}}(\exp X)$. In words, exponentiating a complex matrix and then making it into a real matrix of twice the size is the same as making it a complex matrix of twice the size and then exponentiating the corresponding real matrix.

(d) If $G \subset \mathrm{GL}(m, \mathbb{C})$ is a complex linear group, show that we may use $\iota_{\mathbb{C}}$ to view G as a real linear group. Check that the Lie algebra of G viewed as a real linear group is $\iota_{\mathbb{C}}$ applied to $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$.

The previous exercise allows us to view all complex linear groups as real linear groups. For this reason, we will, from now on, only study real linear groups, and we will refer to them simply as linear groups in an effort to simplify our terminology.

Now, let us return to the setting of Section 1.1.

Proposition 1.11. *Let* $X \subset \mathbb{R}^n$ *be a smooth submanifold. Let* G *be a linear group of symmetries acting transitively on* X*, and let* $\mathfrak{g}$ *denote the Lie algebra of* G*. Fix* $x \in X$*, let* $G_x \subset G$ *denote the stabilizer of* x *in* G*, and let* $\mathfrak{g}_x \subset \mathfrak{g}$ *denote its Lie algebra. Then there is a natural isomorphism*

$$\mathfrak{g}/\mathfrak{g}_x \xrightarrow{\sim} T_xX. \tag{1}$$

Exercise 1.12. (a) First define a smooth map $p_x \colon G \to X$ by $g \mapsto g{\cdot}x$. Show that p is a smooth submersion at $x = e$.

(Hint: Suppose not. Then p must map an open neighborhood $e \in U \subset G$ into a closed, codimension one submanifold of X. Now, show that G may be written as a countable union of open sets of the form gU for some $g \in G$. Then conclude that X must be a countable union of codimension one submanifolds of X. This is a contradiction.)

(b) Check that the derivative of p_x is the surjection $\mathfrak{g} \to T_xX$ by

$$X \mapsto \frac{d}{dt}\Big|_{t=0} \exp(tX) \cdot x.$$

Check that the kernel of this map is $\mathfrak{g}_x$, and deduce (1).

Next, suppose $G \subset \mathrm{GL}(n, \mathbb{R})$ is a linear group, and suppose $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ is its Lie algebra. Notice that G acts on itself by conjugation

$$c(g)\colon x \mapsto gxg^{-1} \text{ for all } g, x \in G.$$

Differentiating this map at $e \in G$, we obtain a linear map

$$\mathrm{Ad}(g)\colon \mathfrak{g} \simeq T_eG \to \mathfrak{g} \simeq T_eG.$$

Exercise 1.13. (a) Check that $\mathrm{Ad}(g)X = gXg^{-1}$ is simply conjugation by matrices.

(b) Check that $G \to \mathrm{GL}(\mathfrak{g})$ by $g \mapsto \mathrm{Ad}(g)$ is a group homomorphism.

We call $\mathrm{Ad}(g)$ the *adjoint action* of $g \in G$ on $\mathfrak{g}$.

1.3. **Invariant Densities on Compact Homogeneous Spaces.** Let G be a linear group, and let $X \subset \mathbb{R}^n$ be an m-dimensional smooth manifold on which G acts smoothly and transitively. We want to integrate functions f on X in such a way that translation by $g \in G$ does not effect the result, ie

$$\int_X f(x) = \int_X f(g \cdot x).$$

Of course, in order to make this precise, we must integrate against something. In a first course on smooth manifolds, you are usually taught to integrate top-dimensional differential forms. Recall that a top-dimensional differential form $\omega \in \Omega^m(X)$ is a function

$$\omega_x\colon T_xX^{\oplus m} \to \mathbb{C}$$

for each $x \in X$ satisfying

$$\omega_x(AX_1, \dots, AX_m) = \det(A)\omega_x(X_1, \dots, X_m)$$

for all $X_1, \dots, X_m \in T_xX$ and $A \in \mathrm{GL}(T_xX)$. Moreover, ω_x must vary smoothly as a function of $x \in X$. The trouble with integrating top-dimensional forms is that they require a choice of orientation on X. For our applications, it is easier to remove this ambiguity and instead integrate densities on X. A *smooth density* $\nu \in \mathcal{D}(X)$ is a function

$$\nu_x\colon T_xX^{\oplus m} \to \mathbb{C}$$

for each $x \in X$ satisfying

$$\nu_x(AX_1, \ldots, AX_m) = |\det(A)|\omega_x(X_1, \ldots, X_m)$$

for all $X_1, \ldots, X_m \in T_xX$ and $A \in \mathrm{GL}(T_xX)$. Moreover, ν_x must vary smoothly as a function of $x \in X$. For a more systematic treatment of densities and how to integrate them see Chapter 16 of [Lee12]. Alternately, you can use what you know about differential forms and common sense.

Let us get back to our integration question. We wish to find a smooth density $\nu \in \mathcal{D}(X)$ such that

$$\int_X f(x)d\nu = \int_X f(g \cdot x)d\nu. \tag{2}$$

Notice (2) holds if, and only if ν is a G-invariant density on X. Therefore, we must ask whether or not there exists a G-invariant density on X. Let us begin with the special case $X = G$. There is a left action of G on $\mathcal{D}(G)$ by

$$(L_g^*\omega)_x(X_1, \ldots, X_m) := \omega_{g^{-1}x}((L_{g^{-1}})_*X_1, \ldots, (L_{g^{-1}})_*X_m)$$

and there is a right action of G on $\mathcal{D}(G)$ by

$$(R_g^*\omega)_x(X_1, \ldots, X_m) := \omega_{xg}((R_g)_*X_1, \ldots, (R_g)_*X_m).$$

To define a left G-invariant density ν on G, first let us define ν at the identity $e \in G$. In particular, we fix a non-zero $\nu_e \colon T_eG^{\oplus m} \to \mathbb{C}$ satisfying $\nu_e(AX_1, \ldots, AX_m) = |\det A|\nu_e(X_1, \ldots, X_m)$ for all $X_1, \ldots, X_m \in T_eG$ and $A \in \mathrm{End}(T_eG)$. Then we define

$$\nu_x := L_{x^{-1}}^*\nu_e.$$

Question: Is this density also right G-invariant?

If so, then we would have

$$(R_x^*L_x^*\nu_e)(X_1, \ldots, X_m) \overset{?}{=} \nu_e(X_1, \ldots, X_m).$$

The left hand side is equal to

$$\begin{aligned} &\nu_e((L_{x^{-1}})_*(R_x)_*X_1, \ldots, (L_{x^{-1}})_*(R_x)_*X_m) \\ =&\nu_e(\mathrm{Ad}(x^{-1})X_1, \ldots, \mathrm{Ad}(x^{-1})X_m) \\ =&|\det(\mathrm{Ad}(x^{-1}))|\nu_e(X_1, \ldots, X_m). \end{aligned}$$

In particular, there exists a simultaneously left and right G-invariant density $\nu \in \mathcal{D}(G)$ only if $|\det(\mathrm{Ad}(x))| = 1$ for all $x \in G$.

Exercise 1.14. Show that there exists a left and right G-invariant density $\nu \in \mathcal{D}(G)$ if, and only if $|\det(\mathrm{Ad}(x))| = 1$ for all $x \in G$.

Exercise 1.15. (a) Define $\delta(x) := |\det(\mathrm{Ad}(x))|$. We call $\delta\colon G \to \mathbb{R}_+$ the *modular function* of G.
(b) Show that δ is a continuous group homomorphism.
(c) If G is compact, show that the image of δ is compact.
(d) Deduce that $\delta = 1$ if G is a compact linear group. In particular, if G is a compact linear group, then every left G-invariant density is also right G-invariant. Further, a non-zero left and right G-invariant density on G is unique up to multiplication by a scalar.

Now, let us go back to the general case. Assume $X \subset \mathbb{R}^n$ is a smooth manifold with a smooth, transitive G-action we call a. Then G acts on $\mathcal{D}(X)$ by

$$(a(g)^* \cdot \nu)_x(X_1, \dots, X_m) := \nu_{a(g^{-1})x}(a(g^{-1})_*X_1, \dots, a(g^{-1})_*X_m).$$

If $x \in X$, let

$$G_x := \{g \in G \mid g \cdot x = x\}$$

denote the stabilizer of x in G. If $x \in X$ and $g \in G_x$, then

$$\begin{aligned}(a(g)^*\nu_x)(X_1, \dots, X_m) &= \nu_x(a(g^{-1})_*X_1, \dots, a(g^{-1})_*X_m)\\ &= |\det(a(g^{-1})_*)|\nu_x(X_1, \dots, X_m).\end{aligned}$$

By (1), we may identify $T_xX \simeq \mathfrak{g}/\mathfrak{g}_x$.

Exercise 1.16. (a) If $g \in G_x \subset G$, check that the adjoint action, $\mathrm{Ad}(g)$, on $\mathfrak{g}$ preserves $\mathfrak{g}_x$ and descends to an action of g on $\mathfrak{g}/\mathfrak{g}_x$.
(b) Check that the induced action of $\mathrm{Ad}(g)$ for $g \in G_x$ on $\mathfrak{g}/\mathfrak{g}_x$ is the same as the action $a(g)_*$ on $\mathfrak{g}/\mathfrak{g}_x$.
(c) Using part (b) deduce

$$|\det(a(g^{-1})_*)| = \delta_G(g^{-1})/\delta_{G_x}(g^{-1}).$$

where δ_G denotes the modular function for G and δ_{G_x} denote the modular function for G_x. Deduce that if X admits a non-zero G-invariant density, then

$$\delta_G|_{G_x} = \delta_{G_x}$$

for all $x \in X$.
(d) Now, translate by an arbitrary $g \in G$. Check that X admits a non-zero G-invariant density if, and only if for some (equivalently all) $x \in X$, we have

$$\delta_G|_{G_x} = \delta_{G_x}.$$

If this density exists, check that it is unique up to multiplication by a scalar.

Exercise 1.15 and Exercise 1.16 imply the following key fact.

Proposition 1.17. *Suppose $X \subset \mathbb{R}^n$ is a smooth manifold and G is a compact linear group acting smoothly and transitively on X. Then there exists a non-zero density $\nu \in \mathcal{D}(X)$ that is invariant under the action of G. Moreover, this density is unique up to multiplication by a non-zero complex scalar.*

We say that a density $\nu \in \mathcal{D}(X)$ is *positive* if

$$\nu_x(X_1, \ldots, X_m) \in \mathbb{R}_{>0}$$

whenever $X_1, \ldots, X_m \in T_x X$ is a linearly independent set.

Exercise 1.18. (a) If $\nu \in \mathcal{D}(X)$ is non-zero and G-invariant, show that $c \cdot \nu$ is positive for some scalar $c \in \mathbb{C}$.
(b) If $\nu_1, \nu_2 \in \mathcal{D}(X)$ are positive and G-invariant, show $\nu_1 = c \cdot \nu_2$ for some $c \in \mathbb{R}_{>0}$.

Therefore, we may Proposition 1.17 also holds with "non-zero density" replaced by "positive density" and "non-zero complex scalar" replaced by "positive real scalar".

2. Lecture 2: Characters of Compact Linear Groups

2.1. Representations of Compact Linear Groups.

Definition 2.1. A *representation* of a (real) linear group G is a pair (π, V) where

(a) V is a topological complex vector space
(b) $\pi\colon G \times V \to V$ is a continuous group action of G on V.

We usually write $\pi(g)v$ instead of $\pi(g, v)$ for this group action.

Definition 2.2. A representation (π, V) of a (real) linear group G is *unitary* if

(a) V is Hilbert space with a fixed inner product $(\cdot, \cdot)$.
(b) The action of G on V preserves the inner product, ie

$$(\pi(g)v, \pi(g)w) = (v, w)$$

for all $v, w \in V$ and $g \in G$.

Example 2.3. Let $X \subset \mathbb{R}^n$ be a smooth manifold with a smooth, transitive action of a linear group G, and let ν be a smooth, positive G-invariant density on X. Then

$$L^2(X) := \left\{ f\colon X \to \mathbb{C} \text{ measurable} \mid \int_X |f(x)|^2 d\nu < \infty \right\}$$

is a Hilbert space with inner product

$$(f_1, f_2) := \int_X f_1(x)\overline{f_2(x)}d\nu.$$

Since ν is unique up to a positive scalar, $L^2(X)$ is independent of the choice of ν. Further, there is a continuous action of G on $L^2(X)$ by

$$(l_g f)(x) := f(g^{-1} \cdot x).$$

Since ν is G-invariant, one checks that this action preserves the inner product on $L^2(X)$. In particular, $(l, L^2(X))$ is a unitary representation of G.

Definition 2.4. We say that two unitary representations (π_1, V_1) and (π_2, V_2) of a linear group G are *isomorphic* if there exists a Hilbert space isomorphism

$$T\colon V_1 \xrightarrow{\sim} V_2$$

satisfying $T(\pi_1(g)v_1) = \pi_2(g)T(v_1)$.

Definition 2.5. If (π, V) is a unitary representation of a linear group G, then an *invariant subspace* of V is a closed subspace $W \subset V$ such that $\pi(g)w \in W$ for all $g \in G$ and $w \in W$. We say that a unitary representation (π, V) is *irreducible* if the only invariant subspaces of V are $\{0\}$ and V.

Suppose $\{V_\alpha, (\cdot,\cdot)_\alpha\}_{\alpha\in\mathcal{A}}$ is a family of Hilbert spaces with fixed inner products. Then we may form the inner product space

$$\bigoplus_{\alpha\in\mathcal{A}} V_\alpha.$$

This is the collection of sums $\sum_{\alpha\in\mathcal{A}} v_\alpha$ where $v_\alpha \in V_\alpha$ and $v_\alpha = 0$ for all but finitely many α. The inner product on the direct sum is given by

$$\left(\sum_{\alpha\in\mathcal{A}} v_\alpha, \sum_{\alpha\in\mathcal{A}} w_\alpha\right) = \sum_{\alpha\in\mathcal{A}} (v_\alpha, w_\alpha)_\alpha .$$

The completion of this inner product is a Hilbert space which we denote by

$$V := \widehat{\bigoplus}_{\alpha\in\mathcal{A}} V_\alpha.$$

Further, if (π_α, V_α) is a unitary representation of a linear group G, then $\pi := \bigoplus \pi_\alpha$ acts on $\bigoplus_{\alpha\in\mathcal{A}} V_\alpha$ preserving the inner product, and then π extends to an action of G on V. This makes (π, V) into a unitary representation of G.

Proposition 2.6. *Let G be a compact linear group.*

(a) If (π, V) is an irreducible, unitary representation of G, then V is finite dimensional.

(b) Let $\widehat{G}$ denote the collection (isomorphism classes) of irreducible, unitary representations of G. If (π, V) is a unitary representation of G, then there exists a multiplicity $m(\tau, V) \in \mathbb{N}$ for every $(\tau, W_\tau) \in \widehat{G}$ and a G-equivariant isomorphism

$$V \xrightarrow{\sim} \widehat{\bigoplus}_{(\tau, W_\tau) \in \widehat{G}} W_\tau^{\oplus m(\tau, V)}.$$

See for instance Section 3.2 of [Sep07] for proofs. We omit them here.

Exercise 2.7. (a) Suppose (π, V) is a finite dimensional representation of a compact linear group G. Show that (π, V) is a unitary representation of G.

(Hint: Fix an inner product $(\cdot, \cdot)$ on V. Then define a new inner product

$$(v, w)_G := \int_{g \in G} (\pi(g)v, \pi(g)w) d\nu$$

where ν is a left and right G-invariant density on G. Show $(\cdot, \cdot)_G$ is a G-invariant inner product on V.)

(b) If (π, V) is an irreducible, unitary representation of a compact linear group G, show that the G-invariant inner product on V is unique up to multiplication by a positive scalar.

(Hint: Define the *contragradient representation* of V to be V^*, the collection of complex linear functionals on V, with the group action

$$(\pi^*(g) \cdot l)(v) := l(\pi(g^{-1})v).$$

Show that a G-invariant inner product $(\cdot, \cdot)$ gives rise to a G-equivariant, conjugate linear isomorphism $V \to V^*$. Given two G-invariant inner products, one can compose one of these isomorphisms with the inverse of the other to obtain an isomorphism of representations $V \to V$. Show that this isomorphism has to be a scalar times the identity since (π, V) is irreducible. Then work backwards to show that the two inner products must be scalar multiples of one another.)

In the next lecture, we will see how *harmonic analysis* on X is really about decomposing $L^2(X)$ into irreducible unitary representations of G. In order to write down this decomposition explicitly, we will need the notion of a character.

2.2. **Characters of Compact Linear Groups.**

Definition 2.8. Let (π, V) be an irreducible, unitary representation of the compact linear group G. The *character* of (π, V) is defined to be

$$\Theta_\pi(x) := \mathrm{Tr}(\pi(x))$$

for $x \in G$.

Exercise 2.9. (a) Show

$$\Theta_\pi(gxg^{-1}) = \Theta_\pi(x)$$

for all $g, x \in G$.

(b) Show that Θ_π is a smooth function on G.

Define $\mathrm{SU}(2) = \{A \in \mathrm{GL}(2, \mathbb{C}) \mid A^*A = I \ \& \ \det A = 1\}$.

Exercise 2.10. (a) Show

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

(b) Show that the Lie algebra of SU(2) is

$$\begin{aligned} \mathfrak{su}(2) &= \{A \in \mathrm{SU}(2) \mid A^* + A = 0\} \\ &= \left\{ \begin{pmatrix} ia & \beta \\ -\overline{\beta} & -ia \end{pmatrix} \mid a \in \mathbb{R}, \ \beta \in \mathbb{C} \right\}. \end{aligned}$$

Let $\mathrm{Pol}(\mathbb{C}^2)$ denote the collection of polynomials on $\mathbb{C}^2$. More precisely, we can name two variables, x and y, and $\mathrm{Pol}(\mathbb{C}^2) \simeq \mathbb{C}[x, y]$. Let $\mathrm{Pol}_d(\mathbb{C}^2)$ denote the vector space of degree d complex-valued polynomials in the variables x and y. Now, SU(2) acts on $\mathrm{Pol}_d(\mathbb{C}^2)$ by

$$(\pi_d(A) \cdot p)(x, y) := p(A^{-1}x, A^{-1}y).$$

Exercise 2.11. Show that $(\pi_d, \mathrm{Pol}_d(\mathbb{C}^2))$ is a representation of SU(2).

Define

$$T := \left\{ t_\theta := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \subset \mathrm{SU}(2).$$

Observe $T \simeq \mathbb{S}^1$ is isomorphic to the unit circle.

Exercise 2.12. (a) Show that a basis for $\mathrm{Pol}_d(\mathbb{C}^2)$ is

$$\{x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\}.$$

(b) Show $\pi_d(t_\theta) \cdot (x^k y^{d-k}) = e^{i(d-2k)\theta}$. Conclude

$$\begin{aligned} \Theta_{\pi_d}(t_\theta) &= e^{id\theta} + e^{i(d-2)\theta} + \cdots + e^{-id\theta} \\ &= \frac{e^{i(d+1)\theta} - e^{-i(d+1)\theta}}{e^{i\theta} - e^{-i\theta}}. \end{aligned}$$

Exercise 2.13. (a) Show that every element $x \in \mathrm{SU}(2)$ may be written as $x = gt_\theta g^{-1}$ for some $t_\theta \in T$.

Let $\nu_{SU(2)}$ denote the unique positive SU(2) left and right invariant density on SU(2) satisfying

$$\int_{\mathrm{SU}(2)} d\nu_{SU(2)} = 1.$$

Let ν_T denote the unique positive T left and right invariant density on T satisfying

$$\int_T d\nu_T = 1.$$

In coordinate, $\nu_T = \frac{|d\theta|}{2\pi}$.

Exercise 2.14. (a) Identify $\mathfrak{su}(2) \simeq \mathbb{R}^3$ via the coordinates

$$\begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix}.$$

Let SU(2) act on $\mathfrak{su}(2) \simeq \mathbb{R}^3$ by the adjoint action (matrix conjugation). Show that SU(2) acts transitively on the spheres $x^2+y^2+z^2 = r$ for $r > 0$.

(b) Show that the stabilizer of the point $(1,0,0)$ is T. Therefore, we may identify $\mathrm{SU}(2)/T \simeq \mathbb{S}^2 \subset \mathbb{R}^3$. Deduce that there exists an SU(2)-invariant positive density $\nu_{\mathrm{SU}(2)/T}$ on $\mathrm{SU}(2)/T \simeq \mathbb{S}^2$ satisfying

$$\int_{\mathrm{SU}(2)/T} d\nu_{\mathrm{SU}(2)/T} = 1.$$

Proposition 2.15 (Weyl)**.** *For all continuous functions $f \in C(\mathrm{SU}(2))$, we have*

(3)

$$\int_{\mathrm{SU}(2)} f(x) d\nu_{\mathrm{SU}(2)}(x) = \frac{1}{2}\int_T |(e^{i\theta}-e^{-i\theta})|^2 \int_{\mathrm{SU}(2)/T} f(gt_\theta g^{-1}) d\nu_{\mathrm{SU}(2)/T}(g) d\nu_T(\theta).$$

This formula is called *Weyl's integral formula* for $G = \mathrm{SU}(2)$. Define $\mathrm{SU}(2)' := \mathrm{SU}(2) \setminus \{e\}$ and $T' := T \setminus \{e\}$. Then the map

$$c\colon\ \mathrm{SU}(2)/T \times T' \to \mathrm{SU}(2)'$$

by $(g,t) \mapsto gtg^{-1}$ is a 2-1 local diffeomorphism. This explains the $\frac{1}{2}$ in the formula. Then one can pullback the density $\nu_{\mathrm{SU}(2)}$ under c and compute directly that the pullback is equal to $|e^{i\theta} - e^{-i\theta}|^2$ times $\nu_{\mathrm{SU}(2)/T} \times \nu_T$. We omit the details of the calculation. In Lecture 5, we will discuss the generalization of this formula to an arbitrary compact, connected Lie group. The function $e^{i\theta} - e^{-i\theta}$ is especially interesting and important. More on this later.

Exercise 2.16. (a) Check directly that (3) holds for $f(x) = 1$.
(b) Use the Weyl's integral formula (3) to prove

$$\int_{\mathrm{SU}(2)} |\Theta_{\pi_d}|^2 = 1.$$

(c) Use Weyl's integral formula (3) to prove

$$\int_{\mathrm{SU}(2)} \Theta_{\pi_m} \overline{\Theta}_{\pi_n} = 1$$

if $m \neq n$.

The calculations in the above exercise are a special case of a general phenomenon. Let G be a compact linear group, and fix a G-invariant, positive density ν on G satisfying $\int_G d\nu = 1$.

Proposition 2.17 (Schur Orthogonality for Characters)**.** *Suppose* (π_1, V_1) *and* (π_2, V_2) *are irreducible, unitary representations of a compact linear group* G*. Then*

$$\int_G \Theta_{\pi_1} \overline{\Theta}_{\pi_2} d\nu = \begin{cases} 1 \text{ if } \pi_1 \simeq \pi_2 \\ 0 \text{ if } \pi_1 \neq \pi_2. \end{cases} \tag{4}$$

We omit the proof. See for instance Section 4.2 of [Kna05] or Section 3.1 of [Sep07] for proofs.

Exercise 2.18. (a) Assume $\pi_1, \ldots, \pi_k$ are distinct irreducible, unitary representations of G. One can define the character Θ_π of

$$\pi \simeq \bigoplus_{i=1}^{k} \pi_i^{\oplus m_i}$$

in the same way as in the irreducible case. Show $\Theta_\pi = \sum_{i=1}^k m_i \Theta_{\pi_i}$.
(b) Utilize (4) to show

$$\int_G |\Theta_\pi|^2 d\nu = \sum_{i=1}^{k} m_i^2.$$

(c) Use part (b) of this Exercise together with part (b) of Exercise 2.16 to deduce that the representations $(\pi_d, \mathrm{Pol}_d(\mathbb{C}^2))$ of SU(2) are irreducible.

3. Lecture 3: Harmonic Analysis on Compact, Connected Linear Groups

3.1. **Fourier Analysis on $\mathbb{S}^1$.** Before discussing harmonic analysis on compact, connected linear groups, let us discuss Fourier analysis on $\mathbb{S}^1$. We view

$$\mathbb{S}^1 := \left\{ e^{i\theta} \mid 0 \leq \theta < 2\pi \right\} \subset \mathbb{C}^\times.$$

For every $n \in \mathbb{Z}$, we have the one-dimensional representation $(\pi_n, \mathbb{C}_n)$ of $\mathbb{S}^1$ by

$$\pi_n(e^{i\theta}) \cdot z := e^{in\theta} z.$$

Exercise 3.1. Show that every irreducible, unitary representation of $\mathbb{S}^1$ is one-dimensional. (Hint: By Proposition 2.6, we know that every irreducible, unitary representation of $\mathbb{S}^1$ is finite-dimensional. Now, if (π, V) is a finite-dimensional irreducible, unitary representation of $\mathbb{S}^1$ and $\dim V > 1$, show that there must exist $e^{i\theta} \in \mathbb{S}^1$ such that $\pi(e^{i\theta})$ is not a scalar times the identity matrix. Then find a non-trivial eigenspace $\{0\} \neq W \subsetneq V$. Show that W is an invariant subspace, contradicting the irreducibility of (π, V).)

Notice that the character of π_n is

$$\Theta_n(e^{i\theta}) = e^{in\theta}.$$

Next, assume G is a compact linear group. Let $\mathrm{Diff}(G) \subset \mathrm{End}(C^\infty(G))$ denote the algebra of smooth differential operators on G. Notice that G acts on $C^\infty(G)$ on the left and right by

$$(L_g^* f)(x) := f(g^{-1}x),\ (R_g^* f)(x) := f(xg).$$

Next, G acts on $\mathrm{Diff}(G) \subset \mathrm{End}(C^\infty(G))$ on the left and right by

$$((L_g)_* D) f := D(L_{g^{-1}}^* f),\ ((R_g)_* D) f := D(R_{g^{-1}}^* f).$$

We say that a differential operator $D \in \mathrm{Diff}(\mathbb{S}^1)$ is *invariant* if

$$(L_g)_* D = (R_g)_* D = D$$

for all $g \in G$. We denote the algebra of left and right G-invariant differential operators on G by $\mathrm{Diff}(G)^G$.

Exercise 3.2. (a) Show

$$\mathrm{Diff}(\mathbb{S}^1)^{\mathbb{S}^1} \simeq \mathbb{C}\left[\frac{d}{d\theta}\right] \simeq \left\{ \sum_{k=0}^{N} a_k \frac{d^k}{d\theta^k} \mid a_k \in \mathbb{C},\ N \in \mathbb{N} \right\}.$$

(Hint: Observe $\mathrm{Diff}(\mathbb{S}^1) \simeq \left\{\sum_{k=0}^{N} f_k \frac{d^k}{d\theta^k} \mid f_k \in C^\infty(\mathbb{S}^1),\ N \in \mathbb{N}\right\}$. Translate on the left or right by an arbitrary $e^{i\varphi}$ and then inductively apply the differential operator to functions that are "locally" equal to 1, θ, θ^2, etc.)

(b) Show by direct calculation that the characters $\Theta_n = e^{in\theta}$ are all eigenfunctions for every differential operator $D \in \mathrm{Diff}(\mathbb{S}^1)^{\mathbb{S}^1}$.

(c) Suppose (π, V) is an irreducible unitary representation of $\mathbb{S}^1$ with character Θ, and assume that the action of $\mathbb{S}^1$ on V is smooth. Show that Θ is an eigenfunction for $\frac{d}{d\theta}$.

(d) Show that all eigenfunctions Θ of $\frac{d}{d\theta}$ on $\mathbb{S}^1$ are of the form $Ce^{in\theta}$ for some $C \in \mathbb{C}$ and $n \in \mathbb{Z}$.

(e) Deduce that every (smooth) irreducible character of $\mathbb{S}^1$ is equal to Θ_n for some $n \in \mathbb{Z}$.

Theorem 3.3 (Abstract Harmonic Analysis on $\mathbb{S}^1$)**.** *There is an isomorphism of unitary representation of* $\mathbb{S}^1$

$$L^2(\mathbb{S}^1) \xrightarrow{\sim} \widehat{\bigoplus_{n\in\mathbb{Z}}} \mathbb{C}_n. \tag{5}$$

If G is a compact linear group and $f, g \in C(G)$ are two continuous functions, then we define the convolution of f and g to be

$$(f * g)(x) := \int_G f(xy^{-1}) g(y) d\nu(y) = \int_G f(y) g(y^{-1}x) d\nu(y)$$

where ν is a positive density on G satisfying $\int_G d\nu = 1$.

Theorem 3.4 (Fourier Analysis on $\mathbb{S}^1$)**.** *If* $f \in C^\infty(\mathbb{S}^1)$*, then*

$$f = \sum_{n\in\mathbb{Z}} f * \overline{\Theta}_n. \tag{6}$$

The sum converges absolutely.

Exercise 3.5. Show $f * \overline{\Theta}_n = a_n e^{in\theta}$ for some constant $a_n \in \mathbb{C}$. We call a_n the nth Fourier coefficient of f.
(Hint: Differentiate $f * \overline{\Theta}_n$ with respect to $\frac{d}{d\theta}$.)

One observes that the decompositions (5) and (6) diagonalize the action of the algebra $\mathrm{Diff}(\mathbb{S}^1)^{\mathbb{S}^1} \simeq \mathbb{C}\left[\frac{d}{d\theta}\right]$ on $L^2(X)$.

One can find proofs of Theorem 3.4 and Theorem 3.3 in many places. For instance, see Chapter 8 of [Fol07].

3.2. **Harmonic Analysis on Compact Linear Groups.** Let G be a compact linear group. Fix a positive G-invariant density ν on G satisfying $\int_G d\nu = 1$. Recall

$$L^2(G) := \left\{ f\colon G \to \mathbb{C} \text{ measurable} \mid \int_G |f(x)|^2 d\nu(x) < \infty \right\}$$

is a Hilbert space with inner product

$$(f_1, f_2) := \int_G f_1(x)\overline{f_2(x)} d\nu(x).$$

Further, G acts on $L^2(G)$ on the left by

$$(l_g \cdot f)(x) := f(g^{-1}x).$$

And G acts on $L^2(G)$ on the right by

$$(r_g \cdot f)(x) := f(xg).$$

Exercise 3.6. Check that both the left and right action of G on $L^2(G)$ preserve the inner product. More precisely, show

$$(l_g f_1, l_g f_2) = (f_1, f_1) \text{ for all } g \in G,\ f_1, f_2 \in L^2(G)$$

and

$$(r_g f_1, r_g f_2) = (f_1, f_1) \text{ for all } g \in G,\ f_1, f_2 \in L^2(G).$$

The actions l and r commute. Therefore, $(l \times r, L^2(G))$ is a unitary representation of $G \times G$.

Exercise 3.7. Suppose (σ, U) and (τ, W) are two irreducible, unitary representations of G.

(a) Show $(\sigma \otimes \tau, U \otimes W)$ is a unitary representation of $G \times G$.
(b) Show $(\sigma \otimes \tau, U \otimes W)$ is an irreducible representation of $G \times G$. (Hint: Suppose $V \subset U \otimes W$ is a non-trivial, proper invariant subspace. Restrict V to a representation $G \times \{e\} \subset G \times G$ and show $V \simeq U \otimes W_1$ for some $\{0\} \neq W_1 \subsetneq W$. Observe that W_1 is not invariant under $\{e\} \times G$ and deduce a contradiction.)

Exercise 3.8. Show that every irreducible, unitary representation of $G \times G$ is of the form $(\sigma \otimes \tau, U \otimes W)$ for two irreducible, unitary representations (σ, U) and (τ, W) of G.
(Hint: Assume (π, V) is an irreducible, unitary representation of $G \times G$. First, restrict to $G \times \{e\}$ and show $V \simeq U \otimes W$ where U is an irreducible, unitary representation of $G \times \{e\}$ and $G \times \{e\}$ acts trivially on W. Then restrict to $\{e\} \times G$.)

If (π, V) is a unitary representation of G, then the *contragradient representation* is (π^*, V^*) where

$$V^* := \{l\colon V \longrightarrow \mathbb{C} \text{ complex linear}\}$$

is the dual space of V and

$$(\pi^*(g)l)(v) := l(\pi(g^{-1})v).$$

By Proposition 2.6, we know that there is a decomposition of $L^2(G)$ into irreducible, unitary representations of $G \times G$. The precise description is due to Peter and Weyl.

Theorem 3.9 (Abstract Harmonic Analysis on Compact Linear Groups)**.** *There is an isomorphism of unitary representations of* $G \times G$

$$L^2(G) \xrightarrow{\sim} \widehat{\bigoplus}_{(\tau, W_\tau)\in\widehat{G}} W_\tau \otimes W_\tau^*.$$

The Peter-Weyl Theorem describes abstract harmonic analysis on $L^2(G)$. For a proof, see for instance ?? and ??. If $(\tau, W_\tau) \in \widehat{G}$, let $\Theta_\tau(g) := \mathrm{Tr}(\tau(g))$ denote the character of (τ, W_τ).

Theorem 3.10 (Fourier Analysis on Compact Linear Groups)**.** *Let* G *be a compact linear group. If* $f \in C^\infty(G)$*, then*

$$f = \sum_{(\tau, W_\tau)\in\widehat{G}} (\dim W_\tau) f * \overline{\Theta}_\tau.$$

The sum converges absolutely.

For a proof, see for instance ?? of [**?**]. As before, let $\mathrm{Diff}(G)^G$ denote the left and right G-invariant differential operators on G. First, this algebra of differential operators is commutative. Second, for every irreducible character Θ_τ of G, there exists an algebra homomorphism

$$\chi_\tau\colon\ \mathrm{Diff}(G)^G \to \mathbb{C}$$

such that

$$D\Theta_\tau = \chi_\tau(D)\Theta_\tau. \tag{7}$$

Finally, if G is a compact, connected linear group, then the map $\tau \mapsto \chi_\tau$ is injective. In particular, these facts imply that the decomposition (3.10) diagonalizes the action of $\mathrm{Diff}(G)^G$ on $C^\infty(G)$. Unfortunately, we do not have time to prove these facts here.

Let $X \subset \mathbb{R}^n$ be an m dimensional smooth submanifold, and assume that G acts smoothly and transitively on X. Let ν denote a G-invariant density on X with $\int_X d\nu = 1$, and define

$$L^2(X) := \left\{ f\colon X \to \mathbb{C} \text{ measurable} \mid \int_X |f(x)|^2 d\nu(x) < \infty \right\}$$

to be the Hilbert space with inner product

$$(f_1, f_2) := \int_X f_1(x)\overline{f_2(x)}d\nu(x).$$

Notice that G acts on $L^2(X)$ by

$$(g \cdot f)(x) := f(g^{-1} \cdot x).$$

Exercise 3.11. Check that this action of G on $L^2(X)$ defines a unitary representation of G on $L^2(X)$.

Fix $x \in X$, and define

$$H := G_x = \{g \in G \mid g \cdot x = x\}$$

be the stabilizer of x in G. Then the pullback of the map

$$G \to G/H \simeq X$$

yields a map

$$\iota_X \colon L^2(X) \to L^2(G).$$

Exercise 3.12. (a) Let $L^2(G)^H$ denote the collection of L^2-functions on G that are invariant under the right action of H. Show

$$\iota_X(L^2(X)) \subset L^2(G)^H.$$

(b) Notice that $L^2(G)^H$ is a unitary representation of G under the left action of G. Show that ι_X is an isomorphism of $L^2(X)$ onto $L^2(G)^H$ as G-representations.

We may utilize Exercise 3.12 to give an analogue of Theorem 3.9 for the homogeneous space X.

Corollary 3.13 (Abstract Harmonic Analysis on Compact Homogeneous Spaces)**.** *Suppose $X \subset \mathbb{R}^n$ is a smooth submanifold, suppose G is a compact linear group acting smoothly and transitively on X, and write $X \simeq G/H$. Then there is an isomorphism of unitary G-representations*

$$L^2(X) \xrightarrow{\sim} \widehat{\bigoplus_{(\tau, W_\tau) \in \widehat{G}}} W_\tau \otimes (W_\tau^*)^H.$$

Exercise 3.14. Deduce Corollary 3.13 from Theorem 3.9 and Exercise 3.12.

Corollary 3.15 (Fourier Analysis on Compact Homogeneous Spaces)**.** *Suppose $X \subset \mathbb{R}^n$ is a smooth submanifold, suppose G is a compact* *linear group acting smoothly and transitively on X, and write $X \simeq G/H$. If $f \in C^\infty(X) \simeq C^\infty(G)^H$, then*

$$f = \sum_{(\tau, W_\tau) \in \widehat{G}} (\dim W_\tau) f * \Theta_\tau.$$

The sum converges absolutely.

Exercise 3.16. If $f \in C^\infty(X) \simeq C^\infty(G)^H$, show

$$f * \Theta_\tau = \Theta_\tau * f \in C^\infty(G)^H \simeq C^\infty(X).$$

In particular, the Fourier coefficients of $f \in C^\infty(X)$ are functions on X.

3.3. **The Example $G = \mathrm{SU}(2)$ and $X = \mathbb{S}^2$.** Recall $G = \mathrm{SU}(2)$ acts on its Lie algebra $\mathfrak{su}(2) \simeq \mathbb{R}^3$ by the adjoint (conjugation) action. As in Exercise 2.14, we may impose coordinates on $\mathbb{R}^3$ so that the orbits of $G = \mathrm{SU}(2)$ are the spheres $x^2 + y^2 + z^2 = r$. Take the unit sphere $X = \mathbb{S}^2$.

Exercise 3.17. Show that the stabilizer of $(1, 0, 0)$ under the action of $\mathrm{SU}(2)$ on $\mathbb{S}^2$ is $T \simeq \mathbb{S}^1$.

First, let us decompose $L^2(X)$ into irreducible representations of $\mathrm{SU}(2)$ abstractly. Recall from Lecture 2 that we constructed the irreducible representations $(\pi_d, \mathrm{Pol}_d(\mathbb{C}^2))$ of $\mathrm{SU}(2)$. Let us shorten the notation and write $V_d := \mathrm{Pol}_d(\mathbb{C}^2)$. It turns out that every irreducible, unitary representation of $\mathrm{SU}(2)$ is of the form (π_d, V_d) for some $d = 0, 1, 2, \ldots$. Then, by Corollary 3.13, we have an isomorphism of $\mathrm{SU}(2)$-representations

$$L^2(\mathbb{S}^2) \simeq \widehat{\bigoplus_{d \in \mathbb{N} \cup \{0\}}} V_d \otimes (V_d^*)^T.$$

Exercise 3.18. (a) Use Exercise 2.12 to show $(V_d)^T \neq 0$ if, and only if $d \in 2\mathbb{N} \cup \{0\}$ is even. Further, when d is even, show $(V_d)^T$ is a one-dimensional space.

(b) Show that V_d irreducible implies V_d^* is irreducible. Since V_d is the unique irreducible, unitary representation of $\mathrm{SU}(2)$ of dimension d, conclude $V_d \simeq V_d^*$. Conclude $(V_d^*)^T \neq 0$ if, and only if $d \in 2\mathbb{N} \cup \{0\}$. Further, when d is even, conclude that $(V_d^*)^T$ is one-dimensional.

The above exercise implies that there is an abstract isomorphism of unitary G-representations

$$L^2(\mathbb{S}^2) \simeq \widehat{\bigoplus_{d \in 2\mathbb{N} \cup \{0\}}} V_d.$$

Next, we wish to write down the Fourier coefficients of $f \in C^\infty(\mathbb{S}^2)$. We know from Corollary 3.15 that our decomposition looks like

$$f = \sum_{d \in 2\mathbb{N} \cup \{0\}} f * \Theta_d. \tag{8}$$

Recall from Exercise 2.12 that one can write down Θ_d more explicitly as

$$\Theta_d(g t_\theta g^{-1}) = \frac{e^{i(d+1)\theta} - e^{-i(d+1)\theta}}{e^{i\theta} - e^{-i\theta}}$$

for all $g \in G = \mathrm{SU}(2)$.

Finally, let us consider $\mathrm{Diff}(\mathrm{SU}(2))^{\mathrm{SU}(2)}$.

Exercise 3.19. Show that SU(2) has a left and right SU(2)-invariant, Riemannian structure.
(Hint: Fix a positive definite form $B_1(\cdot,\cdot)$ on $\mathfrak{g} \simeq T_eG$. Now, average B_1 over $G = \mathrm{SU}(2)$ against the conjugation action to form an $\mathrm{Ad}(\mathrm{SU}(2))$ invariant positive definite form $B(\cdot,\cdot)$ on $\mathfrak{g} \simeq T_eG$. Translate this form around SU(2) to give a left and right invariant Riemannian structure on SU(2)).

Next, we wish to form the Laplacian $\Delta_{\mathrm{SU}(2)}$ on SU(2) with respect to this Riemannian metric. The Riemannian structure $B(\cdot,\cdot)$ on $\mathfrak{g}$ yields isomorphisms

$$\mathfrak{g} \otimes \mathfrak{g} \simeq \mathfrak{g} \otimes \mathfrak{g}^* \simeq \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g},\mathfrak{g}).$$

These isomorphisms are invariant under all linear transformations that preserve $B(\cdot,\cdot)$. Therefore, the identity element I in $\mathrm{Hom}(\mathfrak{g},\mathfrak{g})$ corresponds to an element in $T \in \mathfrak{g} \otimes \mathfrak{g}$. If X_1, X_2, X_3 is an orthonormal basis of $\mathfrak{g}$, then this element is

$$X_1 \otimes X_1 + X_2 \otimes X_2 + X_3 \otimes X_3.$$

This element is invariant under all transformations that preserve $B(\cdot,\cdot)$. Then there is a map $\mathfrak{g} \otimes \mathfrak{g} \to \mathrm{Diff}(\mathfrak{g})$ by $X \otimes Y \mapsto \frac{d}{dX}\frac{d}{dY}$. The image of $T \in \mathfrak{g} \otimes \mathfrak{g}$ in orthonormal coordinates is

$$\frac{d^2}{dX_1^2} + \frac{d^2}{dX_2^2} + \frac{d^2}{dX_3^2}.$$

Once we translate this over all of SU(2), we obtain the Laplacian $\Delta_{\mathrm{SU}(2)}$, and one sees that this differential operator is invariant under left and right translation by SU(2).

Proposition 3.20. *We have*

$$\mathrm{Diff}(\mathrm{SU}(2))^{\mathrm{SU}(2)} \simeq \mathbb{C}[\Delta_{\mathrm{SU}(2)}].$$

Let $\Delta_{\mathbb{S}^2}$ denote the Laplacian on the sphere $\mathbb{S}^2$. It is the case that $\mathrm{Diff}(\mathbb{S}^2)^{\mathrm{SU}(2)} \simeq \mathbb{C}[\Delta_{\mathbb{S}^2}]$. Moreover, $\Delta_{\mathbb{S}^2}$ acts on each component of the decomposition (8) by a different scalar. Therefore, the decomposition (8) diagonalizes the action of $\Delta_{\mathbb{S}^2}$.

4. Lecture 4: Characters and Coadjoint Orbits

4.1. **Coadjoint Orbits.** Let G be a compact, connected linear group with Lie algebra $\mathfrak{g}$. Recall G acts on $\mathfrak{g}$ by the adjoint (conjugation) action. Then G acts on $\sqrt{-1}\mathfrak{g}^* := \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, i\mathbb{R})$ by the dual action

$$(\mathrm{Ad}^*(g)\cdot\xi)(X) := \xi(\mathrm{Ad}(g^{-1})X).$$

An orbit for this coadjoint action is called a *coadjoint orbit* for G. Let $\mathrm{Pol}(\sqrt{-1}\mathfrak{g}^*)$ denote the collection of polynomials on $\sqrt{-1}\mathfrak{g}^*$. The group G acts on polynomials by

$$(g\cdot p)(\xi) := \xi(\mathrm{Ad}^*(g^{-1})\cdot\xi).$$

Let $\mathrm{Pol}(\sqrt{-1}\mathfrak{g}^*)^G$ denote the collection of G-invariant polynomials on $\sqrt{-1}\mathfrak{g}^*$. If $\xi\in\sqrt{-1}\mathfrak{g}^*$, define

$$\Omega_\xi := \left\{\eta\in\sqrt{-1}\mathfrak{g}^* \mid p(\xi)=p(\eta) \text{ for all } p\in\mathrm{Pol}(\sqrt{-1}\mathfrak{g}^*)^G\right\}.$$

One checks that Ω_ξ is a G-invariant set. In particular, it is a union of coadjoint orbits. In fact, we always have that

$$\Omega_\xi = \mathcal{O}_\xi = \mathrm{Ad}^*(G)\cdot\xi \tag{9}$$

is a single coadjoint orbit.

Next, define

$$\mathrm{ad}^*_X\,\xi := \left.\frac{d}{dt}\right|_{t=0}\mathrm{Ad}^*(\exp(tX))\cdot\xi$$

for $\xi\in\sqrt{-1}\mathfrak{g}^*$. Then

$$T_\xi\mathcal{O}_\xi \simeq \left\{\mathrm{ad}^*_X\,\xi \mid X\in\mathfrak{g}\right\}.$$

The Kirillov-Kostant two form ω on $\mathcal{O}_\xi$ is defined to be

$$\omega_\xi(\mathrm{ad}^*_X\,\xi, \mathrm{ad}^*_Y\,\xi) := \xi([X,Y])$$

where $[X,Y] := XY - YX$ is the bracket operation on the Lie algebra $\mathfrak{g}$. It turns out ω is G-invariant and it gives $\mathcal{O}$ the structure of a symplectic manifold for every $\xi\in\sqrt{-1}\mathfrak{g}^*$. In particular, for every $\xi\in\sqrt{-1}\mathfrak{g}^*$, $\dim\mathcal{O}_\xi = 2m$ is even dimensional. Further,

$$\nu_\xi := \left|\frac{\omega_\xi^{\wedge m}}{2\pi\sqrt{-1}}\right| \tag{10}$$

is a G-invariant density on $\mathcal{O}_\xi$.

Example 4.1. Add example SU(2).

Let V be a real, finite-dimensional vector space, and define

$$\sqrt{-1}V^* := \mathrm{Hom}_{\mathbb{R}}(V, \sqrt{-1}\mathbb{R}).$$

Let $\mathcal{M} \subset \sqrt{-1}V^*$ be a compact, smooth submanifold with a positive, smooth density ν. Define the *Fourier transform* of $\mathcal{M}$ to be

$$\mathcal{F}[\mathcal{M}](X) := \int_{\mathcal{M}} e^{\langle \xi, X\rangle} d\nu(\xi).$$

Exercise 4.2. Show $\mathcal{F}[\mathcal{M}]$ is a smooth function on V.
(Hint: Differentiate under the integral sign).

Exercise 4.3. Suppose there is a compact, connected linear group G acting smoothly and transitively on $\mathcal{M}$. Show that $\mathcal{F}[\mathcal{M}]$ is an $\mathrm{Ad}^*(G)$-invariant smooth function on V.

Exercise 4.4. (a) If $v \in V$, show

$$\frac{d}{dv}\mathcal{F}[\mathcal{M}](X) = \int_{\mathcal{M}} e^{\langle \xi, X\rangle} \langle \xi, v\rangle d\nu(\xi).$$

(b) In the first exercise, we have tacitly identified the vector $v \in V$ with the differential operator $\frac{d}{dv}$ on V and the degree one polynomial $\langle \cdot, v\rangle$ on $\sqrt{-1}V^*$. Show that this identification extends to an isomorphism

$$\mathrm{Pol}(\sqrt{-1}V^*) \xrightarrow{\sim} \mathrm{Diff}_0(V).$$

where $\mathrm{Diff}_0(V)$ denote the collection of constant coefficient differential operators on $\sqrt{-1}V^*$. We denote this map

$$p \mapsto \partial(p).$$

(c) If $p \in \mathrm{Pol}(\sqrt{-1}V^*)$, then

$$(\partial(p)\mathcal{F}[\mathcal{M}])(X) = \int_{\mathcal{M}} e^{\langle \xi, X\rangle} p(\xi) d\nu(\xi).$$

Let us return to the terminology of Section 4.1. Put $\mathcal{M} = \mathcal{O}_\xi$, a coadjoint orbit for a compact linear group G, let ν be the density on $\mathcal{O}_\xi$ defined in (10), and put $\mathfrak{g} = V$. Let $\mathrm{Diff}_0(\mathfrak{g})^G$ denote the algebra of constant coefficient $\mathrm{Ad}(G)$-invariant differential operators on $\mathfrak{g}$. Exercise 4.2 and Exercise 4.3 imply that

$$\mathcal{F}[\mathcal{O}_\xi]$$

is a smooth, $\mathrm{Ad}(G)$-invariant function on $\mathfrak{g}$. Further, Exercise (4.4) implies that there exists an algebra homomorphism

$$\chi_\xi \colon \mathrm{Diff}_0(\mathfrak{g})^G \to \mathbb{C}$$

such that

$$D\mathcal{F}[\mathcal{O}_\xi] = \chi_\xi(D)\mathcal{F}[\mathcal{O}_\xi].$$

Add Example: SU(2).

4.2. **The Kirillov/Harish-Chandra Character Formula.** Let G be a compact linear group, let $\mathfrak{g}$ denote the Lie algebra of G, and let $\exp\colon \mathfrak{g} \to G$ be the exponential map. Let ν_G denote a positive, left and right G-invariant density on G, and let $|dX|$ denote a positive, translation invariant density on $\mathfrak{g}$. One can pullback the density ν_G on G to a density $\exp^* \nu_G$ on $\mathfrak{g}$ defined by

$$(\exp^* \nu_G)(X_1, \ldots, X_n) := \nu_G(\exp_* X_1, \ldots, \exp_* X_n)$$

if $n = \dim \mathfrak{g}$ and $X_1, \ldots, X_n \in \mathfrak{g}$. Since ν_G is a smooth, positive density on G, $\exp^* \nu_G$ is a smooth, positive density on $\mathfrak{g}$. Therefore, there exists a smooth function j_G on $\mathfrak{g}$ such that

$$\exp^* \nu_G = j_G(X)|dX|.$$

We multiply $|dX|$ by a positive constant so that $j_G(0) = 1$.

Exercise 4.5. Show $j_G(X)$ is an $\mathrm{Ad}(G)$-invariant function.

Exercise 4.6. If $G = \mathrm{SU}(2)$, observe

$$\mathfrak{t} = \left\{ X_\theta := \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$$

is the Lie algebra of T. Show that for all $X \in \mathfrak{g} = \mathrm{su}(2)$, there exists $g \in G$ and $\theta \in \mathbb{R}$ such that

$$X = \mathrm{Ad}(g)X_\theta.$$

Example 4.7. If $G = \mathrm{SU}(2)$, then

$$j_G(\mathrm{Ad}(g)X_\theta) = \left(\frac{e^{i\theta} - e^{-i\theta}}{i\theta} \right)^2.$$

Lemma 4.8. *There exists a unique analytic square root $j_G^{1/2}$ of j_G satisfying $j_G^{1/2}(0) = 1$.*

Example 4.9. If $G = \mathrm{SU}(2)$, then

$$j_G^{1/2}(\mathrm{Ad}(g)X_\theta) = \frac{e^{i\theta} - e^{-i\theta}}{i\theta}.$$

Definition 4.10. An *invariant eigenfunction* on G is a smooth function $\Theta \in C^\infty(G)$ satisfying the following two properties.

(1) For all $g, x \in G$, we have

$$\Theta(gxg^{-1}) = \Theta(g).$$

(2) There exists an algebra homomorphism
$$\chi_G\colon \operatorname{Diff}(G)^G \to \mathbb{C}$$
satisfying
$$D \cdot \Theta = \chi_G(D)\Theta$$
for all $D \in \operatorname{Diff}(G)^G$.

Definition 4.11. An *invariant eigenfunction* on $\mathfrak{g}$ is a smooth function $\theta \in C^\infty(\mathfrak{g})$ satisfying the following two properties.

(1) For all $g \in G$ and $X \in \mathfrak{g}$, we have
$$\theta(\operatorname{Ad}(g)X) = \theta(X).$$

(2) There exists an algebra homomorphism
$$\chi_{\mathfrak{g}}\colon \operatorname{Diff}_0(\mathfrak{g})^G \to \mathbb{C}$$
satisfying
$$D \cdot \theta = \chi_{\mathfrak{g}}(D)\theta$$
for all $D \in \operatorname{Diff}_0(\mathfrak{g})^G$.

Exercise 4.12. Construct an isomorphism
$$\phi\colon \operatorname{Diff}_0(\mathfrak{g})^G \to \operatorname{Diff}(G)^G.$$
(Hint: Move the $\operatorname{Ad}(G)$-invariant differential operator on $\mathfrak{g} \simeq T_eG$ around the Lie group G using left or right translation.)

Theorem 4.13 (Harish-Chandra)**.** *Let G be a compact linear group with Lie algebra $\mathfrak{g}$. If Θ is an invariant eigenfunction on G, then*
$$\theta := (\Theta \circ \exp) \cdot j_G^{1/2}$$
is an invariant eigenfunction on $\mathfrak{g}$.

Let (π, V) be a finite dimensional representation of a compact linear group G, and let Θ_π denote the character of π. Then Θ_π is an invariant eigenfunction on G as we saw in Lecture 2. Now, we call
$$\theta_\pi := (\Theta_\pi \circ \exp) \cdot j_G^{1/2}$$
the *Lie algebra analogue of the character of* π. By Theorem 4.13, θ_π is an invariant eigenfunction on $\mathfrak{g}$.

Exercise 4.14. Lie algebra analogue of characters of SU(2).

Theorem 4.15 (Harish-Chandra, Kirillov)**.** *Let G be a compact linear group. For every irreducible unitary representation (π, V) of G, there exists a coadjoint orbit $\mathcal{O}_\pi \subset \sqrt{-1}\mathfrak{g}^*$ such that*
$$\theta_\pi = \mathcal{F}[\mathcal{O}_\pi].$$

Exercise 4.16. SU(2) example

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