The Asymptotics of the Support of Plancherel Measure

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Setting

- G denotes a connected, complex reductive algebraic group
- σ denotes an antiholomorphic involution of G
- $(G^\sigma)_e \subset G_\mathbb{R} \subset G^\sigma$ denotes a real form of G
- $H \subset G$ denotes a Zariski closed, unimodular, σ-stable subgroup
- $H_\mathbb{R} := H^\sigma \cap G_\mathbb{R} \subset G_\mathbb{R}$
- $\mathfrak{g}$, $\mathfrak{g}_\mathbb{R}$, $\mathfrak{h}$, $\mathfrak{h}_\mathbb{R}$ denote the Lie algebras of G, $G_\mathbb{R}$, H, $H_\mathbb{R}$
- $H_r \subset G_\mathbb{R}$ denotes a (not necessarily algebraic) closed subgroup of $G_\mathbb{R}$ with Lie algebra $\mathfrak{h}_r = \mathfrak{h}_\mathbb{R}$
- $X := G/H$, $X_\mathbb{R} := G_\mathbb{R}/H_\mathbb{R}$, $X_r := G_\mathbb{R}/H_r$
- Example: $X_r = \mathrm{GL}(n,\mathbb{R})/(\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(k,\mathbb{Z}))$

Question: Which irreducible, unitary representations of $G_\mathbb{R}$ occur in the decomposition of $L^2(X_r)$ into irreducibles? More precisely, we wish to find the support of the Plancherel measure, $\operatorname{supp} L^2(X_r)$.

Irreducible, Unitary Representations of $G_{\mathbb{R}}$: Notation

- $L \subset G$ is a σ-stable Levi subgroup
- $L_{\mathbb{R}} = L^{\sigma} \cap G_{\mathbb{R}}$
- $\mathfrak{l}$, $\mathfrak{l}_{\mathbb{R}}$ are the Lie algebras of L, $L_{\mathbb{R}}$
- $\sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}},\mathfrak{l}_{\mathbb{R}}])^* := \mathrm{Hom}_{\mathbb{R}}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}},\mathfrak{l}_{\mathbb{R}}],\sqrt{-1}\mathbb{R})$
- If $\xi \in \mathfrak{g}^*$, then $\mathfrak{g}(\xi) \subset \mathfrak{g}$ denotes the stabilizer of ξ in $\mathfrak{g}$
- $\sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}},\mathfrak{l}_{\mathbb{R}}])^*_{\text{reg}} := \left\{\xi \in \sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}},\mathfrak{l}_{\mathbb{R}}])^* \mid \mathfrak{g}(\xi) = \mathfrak{l}\right\}$ is the complement of a finite union of proper vector subspaces in $\sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}},\mathfrak{l}_{\mathbb{R}}])^*$
- $\mathfrak{j} \subset \mathfrak{l}$ is a σ-stable Cartan subalgebra
- $\Delta(\mathfrak{g},\mathfrak{j})$, $\Delta(\mathfrak{l},\mathfrak{j})$ denotes the collection of roots of $\mathfrak{g}$, $\mathfrak{l}$ with respect to $\mathfrak{j}$
- $\Delta^+(\mathfrak{g},\mathfrak{j})$ is a choice of positive roots
- $\Delta^+(\mathfrak{l},\mathfrak{j}) = \Delta^+(\mathfrak{g},\mathfrak{j}) \cap \Delta(\mathfrak{l},\mathfrak{j})$

Irreducible, Unitary Representations of $G_{\mathbb{R}}$

- $\rho_{\mathfrak{l}} := \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{l},\mathfrak{j})}\alpha$
- We say $\lambda \in \sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}},\mathfrak{l}_{\mathbb{R}}])^*_{\text{reg}}$ is in the *good range* if

$$\alpha \in \Delta(\mathfrak{g},\mathfrak{j}) \ \&\ \langle \lambda, \alpha^\vee\rangle \in \mathbb{R}_{>0} \Longrightarrow \langle \lambda + \rho_{\mathfrak{l}}, \alpha^\vee\rangle \in \mathbb{R}_{>0}.$$

- This definition is independent of the choice of Cartan subalgebra $\mathfrak{j} \subset \mathfrak{g}$ and the choice of positive roots $\Delta^+(\mathfrak{g},\mathfrak{j})$
- Fix $\lambda \in \sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}},\mathfrak{l}_{\mathbb{R}}])^*_{\text{reg}}$ in the good range
- $\widetilde{L_{\mathbb{R}}}$ denotes the Duflo double cover of $L_{\mathbb{R}}$
- $\Gamma_\lambda\colon \widetilde{L_{\mathbb{R}}} \to \mathbb{S}^1$ is a one-dimensional, genuine, unitary representation satisfying $d\Gamma_\lambda = \lambda$

Given the data $(\mathfrak{l}_{\mathbb{R}}, \Gamma_\lambda)$, one may construct an irreducible, unitary representation $\pi(\mathfrak{l}_{\mathbb{R}}, \Gamma_\lambda)$ of $G_{\mathbb{R}}$ via derived functor modules (Vogan-Zuckerman) or via Ext groups of $G_{\mathbb{R}}$-equivariant, twisted sheaves on a partial flag G/Q where $Q = LN$ (Kashiwara-Schmid, Wong).

The Momentum Map

If $x \in X$, let G_x denote the stabilizer of x in G. Let $\mathfrak{g}_x$ denote the Lie algebra of G_x. The momentum map

$$\mu\colon T^*X \to \mathfrak{g}^*$$

is defined by

$$(x, \xi) \mapsto \xi \in T^*_x X \simeq (\mathfrak{g}/\mathfrak{g}_x)^* \hookrightarrow \mathfrak{g}^*.$$

Theorem (Knop)

If H is unimodular and $X \simeq G/H$, then there exists a Levi subalgebra $\mathfrak{l}^X \subset \mathfrak{g}$ and a dual torus $\mathfrak{a}^*_X \subset (\mathfrak{l}^X/[\mathfrak{l}^X, \mathfrak{l}^X])^*$ satisfying $Z_{\mathfrak{g}}(\mathfrak{a}^*_X) = \mathfrak{l}^X$, unique up to simultaneous G-conjugacy, such that

$$\overline{\mu(T^*X)} = \overline{\mathrm{Ad}^*(G) \cdot \mathfrak{a}^*_X}.$$

Define $d_X := \dim_{\mathbb{C}} \mathfrak{a}^*_X$

The Asymptotic Cone

- W denotes a real, finite-dimensional vector space
- $SW := (W \setminus \{0\})/\{\xi \sim t\xi, t \in \mathbb{R}_{>0}\}$
- $p\colon W \setminus \{0\} \to SW$ denotes the projection
- $\overline{W} = W \sqcup SW$ denotes the compactification of W with a "sphere at infinity"
- If $\Omega \subset W$ is a subset, let $\overline{\Omega} \subset \overline{W}$ denote its closure in $\overline{W}$

Definition

The *asymptotic cone* of a subset $\Omega \subset W$ is defined to be

$$\mathrm{AC}(\Omega) := p^{-1}(\overline{\Omega} \cap SW) \cup \{0\}$$

First Theorem

If $\mathfrak{l}_1, \mathfrak{l}_2 \subset \mathfrak{g}$ are two Levi subalgebras, we write $\mathfrak{l}_1 \sim \mathfrak{l}_2$ if there exists $g \in G$ such that $\mathrm{Ad}(g)\mathfrak{l}_1 = \mathfrak{l}_2$

Theorem (joint with Yoshiki Oshima)

Let $\mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ be a real Levi subalgebra with complexification $\mathfrak{l} \sim \mathfrak{l}^X$. Then

$$\mathrm{AC}\left(\bigcup_{\substack{\pi(\mathfrak{l}_{\mathbb{R}},\Gamma_\lambda)\in \mathrm{supp}\, L^2(X_r) \\ \lambda \textit{ in good range}}} \lambda\right) \cap \sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}}, \mathfrak{l}_{\mathbb{R}}])^*_{reg}$$
$$= \overline{\mu(\sqrt{-1}T^*X_r)} \cap \sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}}, \mathfrak{l}_{\mathbb{R}}])^*_{reg}$$

Further, we show that "most" of supp $L^2(X_r)$ consists of representations of the form $\pi(\mathfrak{l}_{\mathbb{R}}, \Gamma_\lambda)$ with $\mathfrak{l} \sim \mathfrak{l}^X$ (precise statement in Theorem 2).

First Theorem: Remarks

- For every real form $\mathfrak{l}_{\mathbb{R}}$, we either have
$$d_X = \dim \overline{\mu(\sqrt{-1}T^*X_r)} \cap \sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}}, \mathfrak{l}_{\mathbb{R}}])^*_{\text{reg}}$$
$$\text{or} \qquad \emptyset = \overline{\mu(\sqrt{-1}T^*X_r)} \cap \sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}}, \mathfrak{l}_{\mathbb{R}}])^*_{\text{reg}}$$
and the first case occurs for some $\mathfrak{l}_{\mathbb{R}}$.
- In the first case, $\overline{\mu(\sqrt{-1}T^*X_r)} \cap \sqrt{-1}(\mathfrak{l}_{\mathbb{R}}/[\mathfrak{l}_{\mathbb{R}}, \mathfrak{l}_{\mathbb{R}}])^*_{\text{reg}}$ contains an open subset of $(\mathfrak{a}_X^*)_{\mathbb{R}}$. Here $(\mathfrak{a}_X^*)_{\mathbb{R}}$ denotes the real points of a G-conjugate of $\mathfrak{a}_X^*$ defined over $\mathbb{R}$.
- In the work of Sakellaridis-Venkatesh, $\mathfrak{a}_X^*$ is the maximal torus of the Lie algebra of a dual group of X, and the conjectural Plancherel formula for $X_{\mathbb{R}}$ involves parameters closely related to real forms of $\mathfrak{a}_X^*$.
- The work of Delorme-Knop-Krötz-Schlichtkrull determines the support of the Plancherel measure for a spherical variety up to the twisted discrete spectrum. This reduction involves parameters closely related to the split part of a real form of $\mathfrak{a}_X^*$.

An Example

$X_r = \mathrm{GL}(n,\mathbb{R})/(\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(k,\mathbb{Z}))$

- If $2m+1 \leq n$, then $\mathfrak{l}^X$ is a Cartan subalgebra. Hence, Theorem 1 shows the existence of many tempered (for $G_\mathbb{R}$) representations of $G_\mathbb{R}$ occurring in the Plancherel formula for $L^2(X_r)$.
- If $2m+1 > n$, then

$$\mathfrak{l}^X \simeq \mathfrak{gl}(2m-n,\mathbb{C}) \times \mathfrak{gl}(1,\mathbb{C})^{\times(2n-2m)}.$$

The real forms of (conjugates of) $\mathfrak{l}^X$ are

$$\mathfrak{l}^k_\mathbb{R} = \mathfrak{gl}(2m-n,\mathbb{R}) \times (\mathfrak{gl}(1,\mathbb{R}) \times \mathfrak{o}(2,\mathbb{R}))^{\times k} \times \mathfrak{gl}(1,\mathbb{R})^{\times 2(n-m-k)}$$

for $0 \leq k \leq n-m$.

An Example

We may form the larger real Levi subgroup

$$(L_{\mathbb{R}}^k)' = \mathrm{GL}(2m-n,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})^{\times k} \times \mathrm{GL}(1,\mathbb{R})^{\times 2(n-m-k)}.$$

We let $(P_{\mathbb{R}}^k)'$ be a real parabolic with Levi factor $(L_{\mathbb{R}}^k)'$. Let ϵ denote either the trivial representation or the sign of the determinant of $\mathrm{GL}(2m-n,\mathbb{R})$. Let $\sigma_{\eta,s}^{\epsilon}$ be the relative discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with continuous parameter η, discrete parameter s, and parity parameter $\epsilon \in \{0,1\}$. Let τ_{ξ}^{δ} be the one dimensional representation of $\mathrm{GL}(1,\mathbb{R})$ with continuous parameter ξ and parity parameter $\delta \in \{0,1\}$. Then unitary representations of the form

$$\mathrm{Ind}_{(P_{\mathbb{R}}^k)'}^{\mathrm{GL}(n,\mathbb{R})}(\epsilon \boxtimes \sigma_{\eta_1,s_1}^{\epsilon_1} \boxtimes \cdots \boxtimes \sigma_{\eta_k,s_k}^{\epsilon_k} \boxtimes \tau_{\xi_1}^{\delta_1} \boxtimes \cdots \boxtimes \tau_{\xi_{2(n-m-k)}}^{\delta_{2(n-m-k)}})$$

occur in the Plancherel formula for

$$L^2(\mathrm{GL}(n,\mathbb{R})/(\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(k,\mathbb{Z}))).$$

Moreover, we "asymptotically obtain all such representations".

Second Theorem: Notation

We borrow notation from Adams-van Leuwen-Trapa-Vogan. A Langlands parameter is a triple $\Gamma = (J_{\mathbb{R}}, \gamma, R^+_{\sqrt{-1}\mathbb{R}})$ where

- $J_{\mathbb{R}} \subset G_{\mathbb{R}}$ is a Cartan subgroup
- γ is a level one character of the ρ_{abs} double cover of $J_{\mathbb{R}}$
- $R^+_{\sqrt{-1}\mathbb{R}}$ is a choice of positive roots among the collection of imaginary roots for $\mathfrak{g}_{\mathbb{R}}$ with respect to $\mathfrak{j}_{\mathbb{R}}$ for which $d\gamma \in \mathfrak{j}^*$ is weakly dominant

These parameters must satisfy a few additional conditions. To every Langlands parameter Γ, we associate a (possibly nonunitary) irreducible representation $J(\Gamma)$. All irreducible, unitary representations of $G_{\mathbb{R}}$ may be expressed in this form.

Second Theorem

Denote by $\widehat{G}_{\mathbb{R}}^{\mathfrak{l}^X}$ the collection of irreducible, unitary representations of $G_{\mathbb{R}}$ of the form $\pi(\mathfrak{l}_{\mathbb{R}}, \Gamma_\lambda)$ such that $\mathfrak{l}_{\mathbb{R}}$ has complexification $\mathfrak{l} \sim \mathfrak{l}^X$ and λ is in the good range.

Theorem (joint with Yoshiki Oshima)

Fix a Cartan subgroup $J_{\mathbb{R}} \subset G_{\mathbb{R}}$. There exists an algebraic subvariety $V_{X_r,sing} \subset \sqrt{-1}\mathfrak{j}_{\mathbb{R}}^*$ such that

$$\dim V_{X_r,sing} < d_X$$

and

$$\mathrm{AC}\left(\bigcup_{\substack{J(J_{\mathbb{R}},\gamma,R^+_{\sqrt{-1}\mathbb{R}}) \in \mathrm{supp}\, L^2(X_r) \\ J(J_{\mathbb{R}},\gamma,R^+_{\sqrt{-1}\mathbb{R}}) \notin \widehat{G}_{\mathbb{R}}^{\mathfrak{l}^X}}} d\gamma\right) \subset V_{X_r,sing}.$$

Elliptic Elements and Discrete Spectrum

Let $\sqrt{-1}(\mathfrak{g}_{\mathbb{R}})^*_{\text{ell}} \subset \sqrt{-1}\mathfrak{g}^*_{\mathbb{R}}$ denote the subset of elliptic elements.

Theorem (joint with Yoshiki Oshima)

If

$$\mu(\sqrt{-1}T^*X_r) \cap \sqrt{-1}(\mathfrak{g}^*_{\mathbb{R}})_{ell}$$

contains a nonempty open subset of $\mu(\sqrt{-1}T^*X_r)$*, then there exist infinitely many distinct irreducible, unitary representations* (π, V) *such that*

$$\operatorname{Hom}_{G_{\mathbb{R}}}(V, L^2(X_r)) \neq \{0\}.$$

- Example: $L^2(\mathrm{Sp}(2n,\mathbb{R})/(\mathrm{Sp}(2m,\mathbb{R}) \times \mathrm{Sp}(2k,\mathbb{Z}))$ has nonempty discrete spectrum for all $m + k \leq n$.
- In the special case of spherical varieties, this result was obtained by Delorme-Knop-Krötz-Schlichtkrull